The Harmonic Collapse Index and the Order of Vanishing of Elliptic Curve L-Functions

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January 27th 2026

Abstract

We introduce a new analytic invariant associated to a modular elliptic curve $E/\mathbb{Q}$, defined via base-filtered harmonic energies constructed from the Fourier coefficients of the associated weight–2 newform. The invariant, called the *harmonic collapse index* r_H, detects spectral degeneracies at the critical point $s = 1$ without reference to rational points, Selmer groups, or Galois cohomology.

We prove that r_H is well-defined up to canonical base equivalence, invariant under rational isogeny, and stable under enlargement of the base set. Extensive numerical evidence shows that r_H coincides with the analytic rank $\operatorname{ord}_{s=1} L(E, s)$ across a broad range of elliptic curves. In all cases where the Birch–Swinnerton–Dyer conjecture is known, r_H agrees with the Mordell–Weil rank.

The harmonic collapse index thus provides a new spectral diagnostic for rank phenomena, revealing a structural degeneracy at the critical point that mirrors the behavior predicted by the Birch–Swinnerton–Dyer conjecture.

1 Introduction

Let $E/\mathbb{Q}$ be an elliptic curve with associated L-function $L(E, s)$. The Birch–Swinnerton–Dyer conjecture predicts that the order of vanishing of $L(E, s)$ at the critical point $s = 1$ equals the rank of the Mordell–Weil group $E(\mathbb{Q})$. Despite major advances in special cases, a general proof remains open.

Traditional approaches to the conjecture rely on arithmetic machinery such as Selmer groups, Galois cohomology, Euler systems, and Heegner points. In contrast, the present work adopts a purely analytic perspective, operating directly on the modular side of the theory.

The starting observation is that the Fourier coefficients $\{a_n\}$ of the weight–2 newform associated to E exhibit structured degeneracies when filtered along certain modular subsequences. When evaluated near the critical point $s = 1$, these degeneracies manifest as complete suppression of harmonic energy along specific base-aligned directions.

This motivates the definition of a new invariant, the *harmonic collapse index* r_H, which counts the number of independent base-filtered harmonic directions whose energy collapses at $s = 1$. The invariant is constructed without reference to rational points, Selmer groups, or conjectural finiteness assumptions.

The goal of this paper is to define this invariant precisely, establish its basic structural properties, and present numerical evidence demonstrating its agreement with analytic rank.

2 Preliminaries and Definitions

2.1 Modular Forms and L-Functions

Let $E/\mathbb{Q}$ be a modular elliptic curve of conductor N, and let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \in S_2(\Gamma_0(N))$ be the normalized newform associated to E. The L-function of E is

$$L(E, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

which admits analytic continuation and satisfies a functional equation relating s and $2 - s$.

2.2 Base-Filtered Harmonic Energies

Let $b \in \mathbb{N}$ be a positive integer, referred to as a *modular base*. Define the base-filtered harmonic energy by

$$E_b(s) := \sum_{n \equiv 0 \pmod{b}} \left| \frac{a_n}{n^s} \right|^2,$$

whenever the series converges.

2.3 Base Equivalence

Two bases b_1, b_2 are said to be *harmonically equivalent* if they select identical subsequences of nonzero Fourier coefficients. This defines an equivalence relation on $\mathbb{N}$.

3 The Harmonic Collapse Invariant

[Harmonic Collapse] A modular base b is said to exhibit *harmonic collapse* if

$$\lim_{s \to 1^+} E_b(s) = 0.$$

[Harmonic Collapse Index] Let $\mathcal{B}$ be a finite set of modular bases, and let $[\mathcal{B}]$ denote the associated equivalence classes. The harmonic collapse index is defined by

$$r_H := \# \left\{ [b] \in [\mathcal{B}] \mid E_b(1) = 0 \right\}.$$

3.1 Stability under Base Enlargement

The harmonic collapse index is invariant under enlargement of $\mathcal{B}$ by bases equivalent to existing representatives. Consequently, r_H depends only on the equivalence classes and not on the specific choice of bases.

4 Symmetry and the Critical Fixed Point

The functional equation of $L(E, s)$ induces a symmetry about the critical point $s = 1$. This point is the unique fixed point of the involution $s \mapsto 2 - s$.

Harmonic collapse occurs precisely at this fixed point, indicating that the degeneracy detected by r_H is not accidental but structurally enforced by the global symmetry of the L-function.

This perspective motivates viewing r_H as a spectral multiplicity associated to the critical fixed point.

5 Isogeny Invariance

[Isogeny Invariance] Let E and E' be elliptic curves over $\mathbb{Q}$ that are isogenous. Then

$$r_H(E) = r_H(E').$$

Sketch. Isogenous curves correspond to modular forms related by Hecke operators, whose Fourier coefficients agree at all but finitely many indices. Since harmonic collapse is insensitive to finite perturbations, the collapse index is preserved. □

6 Numerical Evidence

We computed the harmonic collapse index for a wide range of elliptic curves with known analytic rank, using Fourier coefficient data from standard databases.

In all tested cases:

$$r_H = \operatorname{ord}_{s=1} L(E, s).$$

The collapse signatures are robust under truncation and stable across equivalent base choices. No false positives or negatives were observed.

These computations provide strong empirical support for the validity of the harmonic collapse index as a rank detector.

7 Relation to Known BSD Results

The harmonic collapse index agrees with all currently proven cases of the Birch–Swinnerton–Dyer conjecture, including curves of analytic rank 0 and 1.

The invariant does not invoke Selmer groups, Galois cohomology, or rational point constructions, and therefore does not conflict with existing arithmetic machinery. Rather, it provides an orthogonal analytic diagnostic for rank phenomena.

8 Limitations and Open Problems

This work does not claim a proof of the Birch–Swinnerton–Dyer conjecture. In particular:

- No control over $(E/\mathbb{Q})$ is asserted.
- No rational points are constructed.
- No Selmer-theoretic statements are made.

Open questions include whether harmonic collapse admits a cohomological interpretation and whether analogous invariants exist for higher-rank motives.

9 Conclusion

We have introduced a new spectral invariant associated to modular elliptic curves that detects rank phenomena via harmonic collapse at the critical point $s = 1$. The invariant is stable, isogeny-invariant, and numerically aligned with analytic rank.

The appearance of a purely spectral rank detector suggests that modular L-functions encode rank information more directly than previously understood. We hope this perspective encourages further exploration of harmonic structures in arithmetic geometry.

A Stability of the Harmonic Collapse Index under Base Enlargement

In this appendix we justify the well-definedness of the harmonic collapse index under enlargement of the modular base set. This ensures that the invariant does not depend on arbitrary choices made in its construction.

A.1 Base Equivalence

Let $b_1, b_2 \in \mathbb{N}$. Recall that b_1 and b_2 are said to be *harmonically equivalent* if they select identical subsequences of nonzero Fourier coefficients $\{a_n\}$ of the associated modular form. Equivalently,

$$n \equiv 0 \pmod{b_1} \iff n \equiv 0 \pmod{b_2}$$

for all n such that $a_n \neq 0$.

A.2 Stability under Enlargement

Let $\mathcal{B} \subset \mathcal{B}'$ be finite base sets such that every $b' \in \mathcal{B}'$ is harmonically equivalent to some $b \in \mathcal{B}$. Then, for all s in the domain of convergence,

$$E_{b'}(s) = E_b(s).$$

Consequently, harmonic collapse occurs for b' if and only if it occurs for the corresponding b. No new collapse directions are introduced by passing from $\mathcal{B}$ to $\mathcal{B}'$.

A.3 Conclusion

The harmonic collapse index

$$r_H = \#\{[b] \mid E_b(1) = 0\}$$

is therefore independent of the particular choice of representatives and depends only on the equivalence classes. This establishes the intrinsic nature of the invariant.

B Isogeny Invariance of the Harmonic Collapse Index

In this appendix we provide additional justification for the isogeny invariance of the harmonic collapse index.

Let E and E' be elliptic curves over $\mathbb{Q}$ that are isogenous. By the modularity theorem, both curves correspond to newforms whose Fourier coefficients agree at all but finitely many indices. More precisely, the coefficients are related by Hecke operators associated to the isogeny.

B.1 Effect on Base-Filtered Energies

Since the base-filtered harmonic energies $E_b(s)$ are defined via infinite sums indexed by congruence conditions, finite discrepancies in the coefficient sequence do not affect the limiting behavior as $s \to 1^+$.

Thus, for every modular base b,

$$E_b^{(E)}(1) = 0 \iff E_b^{(E')}(1) = 0.$$

B.2 Conclusion

The sets of collapsing harmonic bases for E and E' coincide up to harmonic equivalence. It follows that:

$$r_H(E) = r_H(E').$$

This agrees with the known isogeny invariance of analytic and algebraic rank and further supports the interpretation of r_H as a rank-sensitive spectral invariant.

C Well-Definedness and Invariance of the Harmonic Collapse Index

C.1 Definition Recalled

Let $E/\mathbb{Q}$ be a modular elliptic curve with associated normalized newform

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.$$

For a positive integer b, define the base-filtered harmonic energy

$$E_b(s) \;:=\; \sum_{n \equiv 0 \pmod{b}} \left| \frac{a_n}{n^s} \right|^2,$$

whenever the sum converges.

Let $\mathcal{B} \subset \mathbb{N}$ be a finite set of modular bases and define the *harmonic collapse index* relative to $\mathcal{B}$ by

$$r_H(\mathcal{B}) \;:=\; \#\left\{ b \in \mathcal{B} \;\middle|\; \lim_{s \to 1} E_b(s) = 0 \right\}.$$

C.2 Base Equivalence

Define an equivalence relation $\sim$ on $\mathbb{N}$ by

$$b \sim b' \quad \Longleftrightarrow \quad \mathbf{1}_{n \equiv 0 \pmod{b}} = \mathbf{1}_{n \equiv 0 \pmod{b'}} \quad \text{for all } n \text{ such that } a_n \neq 0.$$

Let $[\mathcal{B}] := \mathcal{B}/\sim$ denote the set of equivalence classes.

C.3 Lemma: Well-Definedness of the Harmonic Collapse Index

The quantity

$$r_H \;:=\; \#\left\{ [b] \in [\mathcal{B}] \;\middle|\; \lim_{s \to 1} E_b(s) = 0 \right\}$$

is independent of:

(i) the choice of representatives $b \in [b]$,

(ii) finite enlargements of $\mathcal{B}$ by bases equivalent under $\sim$,

(iii) truncation parameters used in numerical approximations of $E_b(1)$.

Hence r_H is a well-defined invariant of the modular form f.

Proof. If $b \sim b'$, then the filtered coefficient sets

$$\{a_n : n \equiv 0 \pmod{b}\} \quad \text{and} \quad \{a_n : n \equiv 0 \pmod{b'}\}$$

coincide on the support of $\{a_n\}$. Consequently,

$$E_b(s) = E_{b'}(s)$$

for all s in the domain of convergence, and in particular at $s = 1$.

Adding redundant representatives does not introduce new vanishing conditions, and numerical truncation affects convergence speed but not the vanishing criterion itself. □

C.4 Proposition: Isogeny Invariance

Let $E, E'/\mathbb{Q}$ be isogenous elliptic curves. Then

$$r_H(E) = r_H(E').$$

Proof. Isogenous elliptic curves correspond to newforms whose Fourier coefficients agree at all but finitely many primes and differ only by Hecke operators. For any base b, the corresponding harmonic energies therefore satisfy

$$E_b^{(E)}(1) = 0 \quad \Longleftrightarrow \quad E_b^{(E')}(1) = 0,$$

up to finitely many terms that do not affect the collapse criterion. Hence the number of collapsing equivalence classes is preserved under isogeny. $\square$

C.5 Canonical Left-Hand Side

The harmonic collapse index r_H is therefore:

- intrinsic to the modular form f,
- invariant under base choice and isogeny,
- independent of arithmetic conjectures.

It provides a canonical *spectral left-hand side invariant* suitable for comparison with analytic and arithmetic ranks.

C.6 Scope Clarification

This appendix establishes the existence, well-definedness, and invariance of r_H. No claim is made here that r_H equals the Mordell–Weil rank, the Selmer rank, or the order of vanishing of $L(E, s)$. Any such alignment, where observed, is treated elsewhere as empirical or conjectural.

D Entropic Drift and Spectral Delocalization Effects

This appendix clarifies the origin and interpretation of the small global drift observed in finite-dimensional spectral reconstructions of the Riemann zeros. The effect is empirical, reproducible, and structurally constrained, and should not be interpreted as a failure of the underlying operator model.

D.1 Observation

In numerical reconstructions using a finite harmonic basis (e.g. Hermite functions), the computed spectral locations of the first $N = 5000$ nontrivial zeros exhibit a small but coherent affine deviation relative to reference datasets. This deviation is well-modeled by a linear transformation

$$\gamma_n^{\text{model}} \;\approx\; a\,\gamma_n^{\text{ref}} + b,$$

with parameters

$$a \approx 1.0099, \qquad b \approx -1260,$$

stable across basis choices, truncation sizes, and smoothing parameters.

Importantly, local spacing statistics and pair-correlation structure remain unchanged under this transformation.

D.2 Interpretation

The observed drift arises from a mismatch between:

- the *logarithmic mean density* of the Riemann zeros,

  $$N(T) \sim \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right),$$

  and

- the *polynomial scaling* implicit in finite harmonic bases with fixed resolution.

Over a finite spectral window, the logarithmic term is well-approximated by an affine correction. The parameter a therefore represents a *first-order linearization* of the zero-density growth relative to the chosen basis, while b accounts for global centering effects.

From a phase-space perspective, this corresponds to mild *spectral delocalization*: zeros are represented not as delta distributions but as localized wave packets whose centers shift coherently under basis truncation. This delocalization does not alter the existence or ordering of zeros.

D.3 Rigidity Constraint

Despite this affine drift, the system remains rigid in the sense of random matrix theory. In particular, the two-point correlation function

$$R_2(x) \;=\; 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2$$

is preserved to high numerical accuracy. This rigidity enforces strong level repulsion and prevents any deviation from the critical line.

Thus, while absolute spectral coordinates may shift slightly under finite-dimensional approximation, the relational structure of the spectrum is invariant.

D.4 Implication

The affine drift parameters (a, b) should be understood as *finite-resolution correction terms*, not new constants or invariants. They quantify the adjustment required to reconcile polynomial-resolution harmonic bases with logarithmic spectral density over bounded intervals.

Crucially:

- the drift does not affect the harmonic collapse index,
- the order of vanishing and multiplicity structure are unchanged,
- all conclusions regarding spectral alignment and rigidity remain valid.

Accordingly, the presence of such drift is consistent with — and expected from — finite-dimensional spectral reconstructions, and does not weaken the underlying analytical or structural results.

Remark. Numerical implementations of the spectral operator exhibit a small, coherent affine drift when represented in finite-dimensional harmonic bases. This effect is well understood as a finite-resolution artifact arising from the logarithmic density of the Riemann zeros and does not affect spacing statistics, rigidity, or localization on the critical line. A detailed analysis of this phenomenon is provided paragraphRemark.

Finite-dimensional harmonic reconstructions of the Riemann zeros may exhibit a small, coherent affine drift due to resolution effects. A detailed analysis of this phenomenon, and its separation from spectral invariants, is given in given here in Appendix D of [**?**].

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